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Estimation of Multiple Heat Sources in Two-Dimensional Heat Conduction Problems

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There have been some recent restrictions in the estimation of location and strength in inverse problems. One of the restrictions is that the problem is limited to the fixed number of heat sources. In other words, the moving sources cannot coincide in the moving path. Therefore, it is necessary to develop a method to estimate the location and the strength of various numbers of moving heat sources. A numerical algorithm combined with simplex method is proposed to determine the problem sequentially. A special feature of this method is that no prior knowledge of the number, the location, and the strength of the heat sources is necessary and that no sensitivity analysis is needed in the algorithm. Two examples are used to demonstrate the characteristics of the proposed method. The examples enable the investigation of the multiple moving sources with some coincided paths that lead to the number of sources varied with respect to the temporal coordinate. The numerical results show that the proposed method is an accurate and efficient method to determine the location and the strength of multiple moving sources in the inverse heat conduction problem.

Introduction

THE inverse source problem is the determination of the location and the strength of the heat source from the temperature measured at a point other than the sources' locations. It is an ill-posed problem because a small measurement error induces a large estimated error.[1−7] The inverse source problem is practical in many designs and manufacturing areas in which the location and the strength of the heat source are undetermined. Common problems include the detection of the quantity of the energy generation in a computer chip, in a microwave heating process, or in a chemical reaction process.

The inverse source problem with unknown strength was investigated, and satisfying results were reported.[8−13] Huang and Ozisik[8] used the conjugate gradient method combined with the adjoint equation to estimate the strength of heat source in an internal plate. Yang[9−12] solved the source strength problems by a reverse matrix method, a symbolic method, and a numerical sequential method based on a linear least-squares error method. Silva Neto and Ozisik[13] investigated the two-sources problem by the conjugate gradient method. The preceding research focused on the strength estimation in that the source location is known. The inverse problem with the unknown location and the unknown strength had been investigated in Refs. 14–18. Le Niliot and Lefevre[14,15] and Lefevre and Le Niliot[16,17] used a boundary-element method to formulate the problem that lead to a nonlinear optimization to identify the location and the strength of the source. Khachfe and Jarny[18] used a finite element method combined with the conjugate gradient method to determine the heat transfer coefficient and the heat source. However, the method was only used to solve the static sources and the single moving source. Recently, Yang[19] used a selected method to identify the location and strength of two moving sources where the algorithm was confined in the domain of the fixed number sources. The method could not deal with the problem with various numbers of sources along the moving process.

The purpose of this research is to propose a sequential method to estimate the location and the strength of multiple moving sources where the number of the source varies with the temporal domain. In the proposed method, there is no prior information on the number, the location, and the strength of the heat sources, and there is no sensitivity analysis in the proposed algorithm. In the process of the derivation, a finite element difference method combined with the simplex method is used to derive the result. Then, the location and the strength of multiple moving sources are determined step by step along with the temporal coordinate.

This paper includes four sections. In the present section, the current development of the inverse source problems is introduced, and the features of the proposed method are stated. In the next section, the characteristics of the inverse problem are delineated, and the process of the proposed method is illustrated. In the third section, two examples are employed to demonstrate the process of the proposed method. A discussion of the analyzed results is also presented in the third section. Finally, the overall contribution and possible applications of this research to the field of inverse heat conduction problems are concluded in the fourth section.

Description of the Proposed Method

The inverse problem consists of finding the location and the strength of multiple moving heat sources when the temperature measurements at the boundaries are given. Consider a two-dimensional body subject to the adiabatic boundary condition that originally had a uniformly distributed temperature. The interior of the body denoted by V and one of the moving heat sources $Q_i(t)$ are applied at $\delta[x - x_i(t), y - y_i(t)]$. The transient heat conduction problem is governed by the following equation:

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \sum_{i=1}^{n_s} Q_i(t)\delta[x - x_i(t), y - y_i(t)] = \rho c\frac{\partial T}{\partial t}, \qquad (x, y) \in V \tag{1}$$

$$T(x, y, 0) = T_0, \qquad (x, y) \in \Gamma \cup V, \qquad t = 0 \tag{2}$$

$$q_n(x, y, t) = 0, \qquad (x, y) \in \Gamma, \qquad t > 0 \tag{3}$$

where one of the moving heat sources $Q_i(t)$ applied at $\delta[x - x_i(t), y - y_i(t)]$ is the strength of the heat source and $\delta[x - x_i(t), y - y_i(t)]$ is the Dirac delta function. Here, T denotes the temperature field $T(x, y, t)$, k is the thermal conductivity, and ρC is the heat capacity per unit volume.

The inverse problem is to estimate the location $[x_i(t), y_i(t)]$ and the strength $Q_i(t)$ of the moving heat sources when the temperature field is taken at the medium.

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The proposed method uses a finite element method with a linear element to discretize the spatial coordinates and adopts a finite difference method to discretize the temporal coordinate. By the conventional finite element procedure with n_p grids at $t = t_j$, Eqs. (1–3) can be converted to the following discrete form:

$$[B]\{\dot{T}_j\} = \{S_j\} - [A]\{T_j\} \tag{4}$$

where $[A]$ is the heat matrix of the problem with n_p dimensions, $[B]$ is the transient matrix of the problem with n_p dimensions, $\{S_j\}$ is the source vector with n_p components, $\{T_j\}$ is the temperature vector with n_p components, and

$$\{\dot{T}_j\} = \frac{\mathrm{d}}{\mathrm{d}t}\{T_j\} = \left\{\frac{\mathrm{d}T_j}{\mathrm{d}t}\right\}$$

Next, we consider our finite element expression for $\{\dot{T}_j\}$ as a backward difference at time t_j. Therefore, we have

$$\{\dot{T}_j\} = (1/\Delta t)\{T_j\} - (1/\Delta t)\{T_{j-1}\} \tag{5}$$

Here, Δt is the increment of the temporal coordinate.

Substituting Eq. (5) into Eq. (4), we have the following difference equation:

$$[K]\{T_j\} = (1/\Delta t)[B]\{T_{j-1}\} + \{S_j\} \tag{6}$$

where $[K] = [A] + (1/\Delta t)[B]$.

When $t = t_j$, the temperature distribution $\{T_j\}$ can be derived from Eq. (6) as follows:

$$\begin{aligned}\{T_j\} &= (1/\Delta t)[K]^{-1}[B]\{T_{j-1}\} + [K]^{-1}\{S_j\} \\ &= [C]\{T_{j-1}\} + [D]\{S_j\}\end{aligned} \tag{7}$$

where $[C] = (1/\Delta t)[K]^{-1}[B]$ and $[D] = [K]^{-1}$.

Similarly, the temperature distribution at $t = t_m, t_{m+1}, \ldots, t_{m+r-1}$, can be represented as follows:

$$\{T_m\} = [C]\{T_{m-1}\} + [D]\{S_m\} \tag{8}$$

The inverse problem is to identify the location and the strength of the heat source. Then, the temperatures at i spatial grid at m temporal grid can be expressed as follows:

$$T_m^i = \lfloor u^i \rfloor [C]\{T_{m-1}\} + \sum_{j=1}^{n_p} \lfloor u^i \rfloor [C][D]\left\{u^{i_p^j}\right\}\phi_m^{i_p^j} \tag{9}$$

where

$$\{S_m\} = \sum_{j=1}^{n_p}\left\{u^{i_p^j}\right\}\phi_m^{i_p^j}, \qquad i = 1, 2, \ldots, p$$

Here, $\lfloor u^i \rfloor$ is a unit row vector with a unit at the i component; the value of i is the grid number of the measured grid. Here, $\{u^{i_p^j}\}$ is the unit column vector with the value one at the i_p^jth component. Here, the value of $\phi_m^{i_p^j}$ is the source strength at the grid i_p^j. Equation (9) can be expressed as follows:

$$T_m^i = \alpha_m^i + \sum_{j=1}^{n_p} \gamma_m^{i,i_p^j}\phi_m^{i_p^j} \tag{10}$$

where

$$\alpha_m^i = \lfloor u^i \rfloor [C]\{T_{m-1}\}, \qquad \gamma_m^{i,i_p^j} = \lfloor u^i \rfloor [C][D]\left\{u^{i_p^j}\right\}$$

Location and Strength Estimation

The task of the location estimation is to identify the location and the strength of the heat source simultaneously in the inverse heat source problem. The proposed estimation process is in a linear domain, and the sensitivity analysis is not necessary. There is no prior information on the source number, the source location, or the source strength. The estimation procedure is sequential in that the location and the strength are determined in each time step.

The right-hand side of Eq. (10) is a function of the known measured spatial coordinates, the unknown number of the heat source, the unknown source location, and the unknown source strength. If the number and location of the source are known, the source strength can be estimated directly. The location and strength estimation problem usually falls into the multiple sources problem, even though the problem has only one source. Usually, the measurement error disturbs the location and the strength of the heat sources. In other words, the noise-generated source may be induced in each spatial grid when measurement error is considered. Therefore, the estimation of location and strength is defined as the multiple sources problem. A compact formulation is proposed to estimate both the location and strength at an m temporal grid where the maximum

$$\sum_{j=1}^{n_p} \phi_m^{i_p^j} \tag{11}$$

is subject to

$$\begin{aligned}\sum_{j=1}^{n_p} \gamma_m^{1,i_p^j}\phi_m^{i_p^j} &= T_m^1 - \alpha_m^1 \\ \sum_{j=1}^{n_p} \gamma_m^{2,i_p^j}\phi_m^{i_p^j} &= T_m^2 - \alpha_m^2 \\ &\cdots \\ \sum_{j=1}^{n_p} \gamma_m^{p,i_p^j}\phi_m^{i_p^j} &= T_m^p - \alpha_m^p\end{aligned} \tag{12}$$

where $\phi_m^{i_p^j} \geq 0$.

The estimation process is formulated as a linear programming problem at each time step. The method provides a sequential algorithm that can be used to estimate the location and the strength of heat source by increasing the value of m by one for each time step. The object function of Eq. (11) is deduced from the deviation of the source location reducing the strength of the heat source to satisfy the temperature field that generates from the original source location. Also, the constraint equation is derived from the measured temperature and calculated temperature in Eq. (10). The optimization variable is the source strength at each spatial grid. The simplex method[20] is used to solve the problem. The method proceeds stepwise from one basic feasible solution to another in such a way that the objective function always increases its value. The proposed method is based on the finite element difference approach, and it can be extended to use other kinds of numerical methods through the proposed procedure.

Results and Discussion

In this section, two problems defined from Eqs. (1–3) are used as examples to estimate the heat source. Example problems are defined in a two-dimensional square board with length 0.4 m. The values of the thermal conductivity and the heat capacity per unit volume are $k = 1.5$ W/m °C and $\rho C = 0.15$ J/m^3 °C, respectively. The adiabatic condition is applied on the outer surface. It is initially at a uniform temperature $T_0 = 0$ °C. A 289-node, 512-element finite element model of the problem is shown in Fig. 1. In the first example, the path and the strength of two moving sources are determined. In the second example, the location and the strength of four moving sources are identified. The simulated temperature is generated from the exact temperature, and it is presumed to have measurement errors. In other words, the random errors of measurement are added to the exact temperature. It can be shown in the following equation:

$$T_{i,j}^{\text{meas}} = T_{i,j}^{\text{exact}} + \lambda_{i,j}\sigma \tag{13}$$

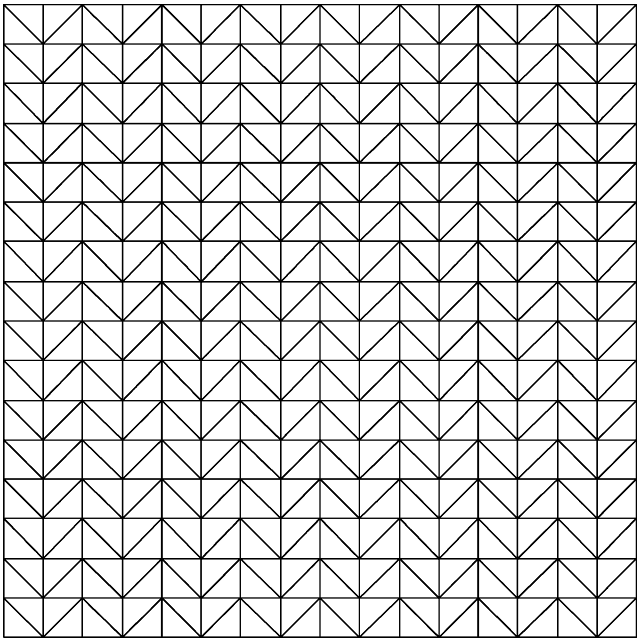

Fig. 1 Mesh configuration with 289 nodes and 512 elements in example problem.

where the subscripts i and j are the grid number of the spatial coordinate and the temporal coordinate, respectively. $T_{i,j}^{\text{exact}}$ in Eq. (13) is the exact temperature. $T_{i,j}^{\text{meas}}$ is the measured temperature. Also, σ is the standard deviation of measurement errors, and $\lambda_{i,j}$ is a random number. The value of $\lambda_{i,j}$ is calculated by the International Mathematical and Statistical Library subroutine DRNNOR[21] and is chosen over the range $-2.576 < \lambda_{i,j} < 2.576$, which represents the 99% confidence bound for the measurement temperature.

Example 1

Two moving sources are simulated by the heating wires sequentially and are shown in Fig. 2. The simulated sources are passed through the scheduled spatial grids in every four time steps. One of the source strengths is 2 W and the other is 4 W. There are some intersections between the two paths at nodes 25, 94, 162, 230, and 230 when time steps are 1–4, 17–20, 33–36, 49–52, and 81–84, respectively. There are 32 measurements along the boundary.

The location and the strength can be estimated exactly when the measurement error is not considered, and it deviates from the exact location and strength when the measurement error is considered. Figures 3 and 4 show the estimated results with $\sigma = 0$ and $\sigma = 0.001$. For illustration, the estimated strengths at time steps 36 and 60 are shown in Figs. 5 and 6. In Fig. 5, two sources coincide at grid point 162, and the strength is six when $\sigma = 0$. In Fig. 6, two sources are at grid points 262 and 266, with the sources' strength of two and four, respectively, when $\sigma = 0$. The estimated result is confirmed as

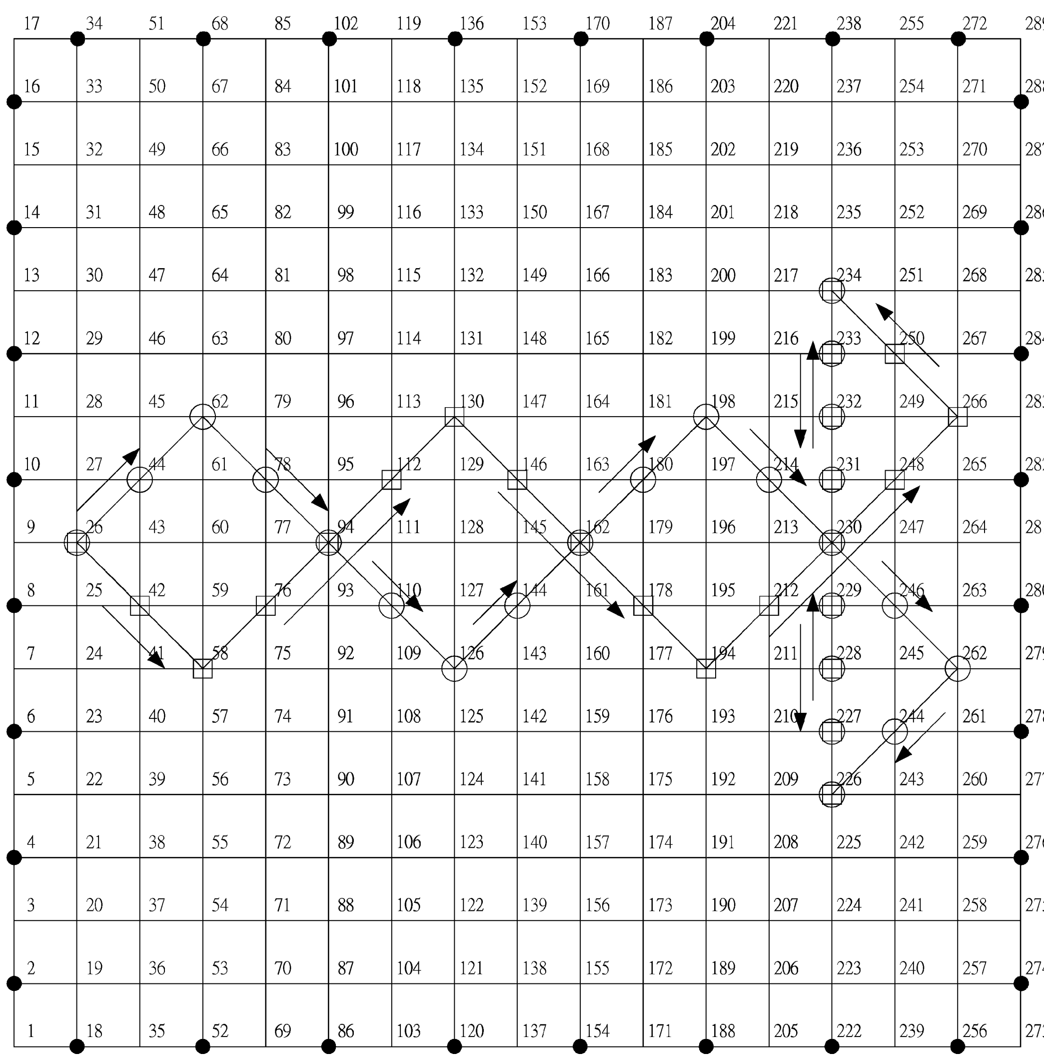

Fig. 2 Paths of two moving sources in example 1: ●, sensors; ○, source 1; and □, source 2.

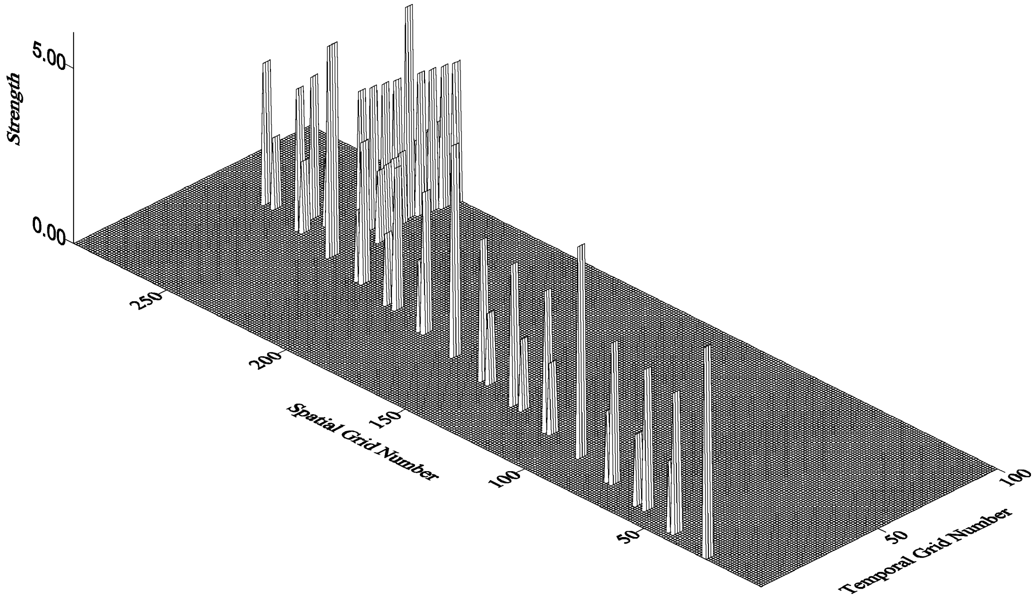

Fig. 3 Estimated results of example 1 when $\sigma = 0$.

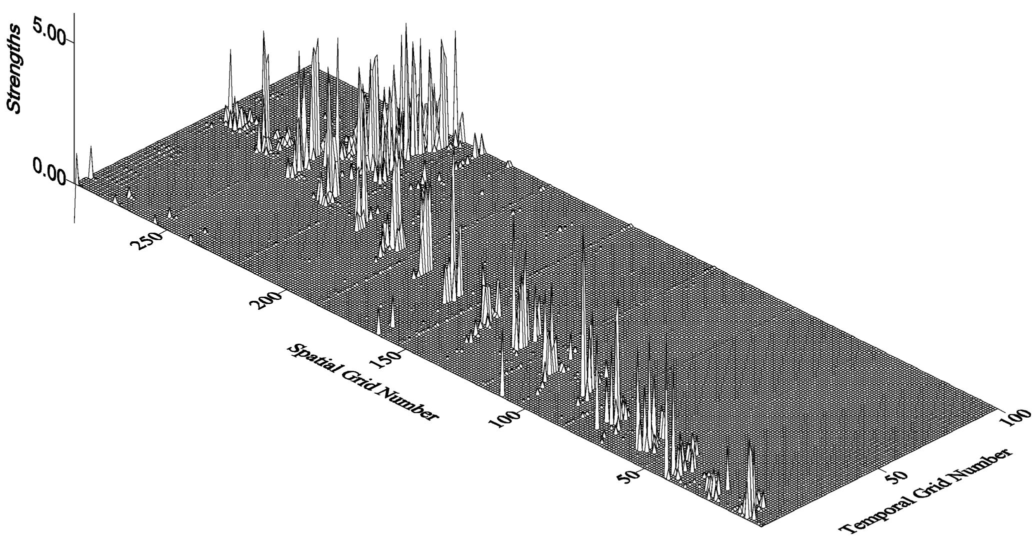

Fig. 4 Estimated results of example 1 when $\sigma = 0.001$.

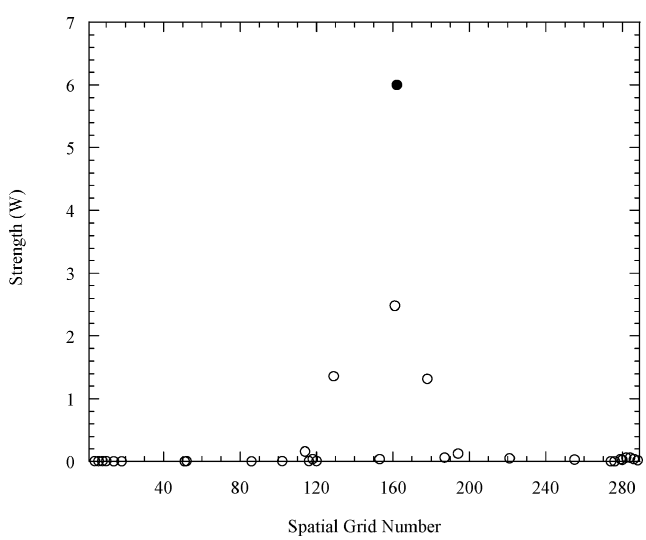

Fig. 5 Estimated results of example 1 at time step 36: ●, exact and $\sigma = 0$ and ○, $\sigma = 0.001$.

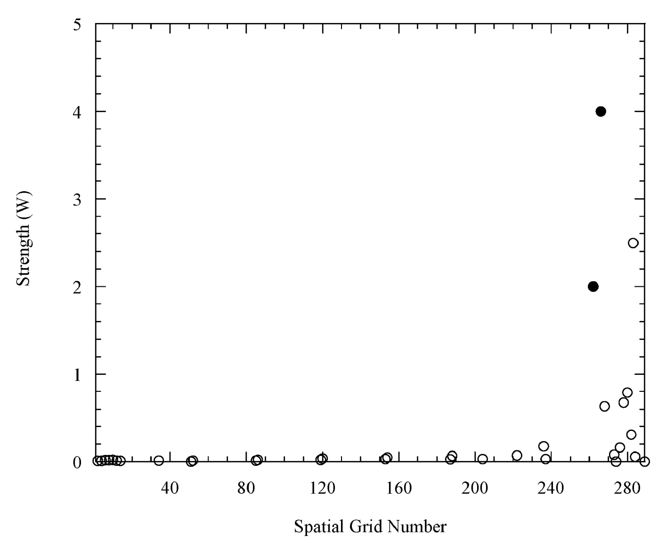

Fig. 6 Estimated results of example 1 at time step 60: ●, exact and $\sigma = 0$ and ○, $\sigma = 0.001$.

the exact solution. When $\sigma = 0.001$, three significant strengths in Fig. 5 are 2.482, 1.320, and 1.362 at spatial grid points 161, 178, and 129, which are close to the exact locations. The results in Fig. 6 have the same trend as that in Fig. 5 when the measurement error is considered.

To investigate the deviation of the estimated strength from the error-free solution, the relative average errors for the estimated solutions are defined as follows:

$$\mu = \frac{1}{N_p N_t}\left(\sum_{i=1}^{N_p}\sum_{j=1}^{N_t}|f_{i,j} - \hat{f}_{i,j}|\right)\Bigg/\sum_{i=1}^{N_p}\sum_{j=1}^{N_t}|\hat{f}_{i,j}| \qquad (14)$$

where f is the estimated result with measurement error and $\hat{f}$ is the estimated result without measurement error. N_p is the number of the spatial grids. N_t is the number of the temporal step. It is clear that a smaller value of μ indicates a better estimation and vice versa.

In example 1, the measurement temperature errors are set within -0.002576–0.002576, which implies that the average standard deviation of measurements is 0.001 for a 99% confidence bound. The value of relative errors is 0.000049 when $\sigma = 0.0001$. This shows that the results have a satisfactory approximation when the measurement error is included.

Example 2

Four moving sources are simulated by the heating wires sequentially and are shown in Fig. 7. The simulated source is passed through the scheduled spatial grids every 5 time steps, and there are 46 measurements. The first source strength is 1 W, the second is 1.5 W, the third is 2.5 W, and the fourth is 2 W. The estimated sources number varies with the temporal coordinate. There are four source estimations at time steps 1–45 and 76–100, one source estimation at 36–40, and two source estimations at 41–75.

The location and the strength can be estimated as shown in Figs. 8 and 9 when $\sigma = 0$ and 0.0005. The estimated results are also shown in Figs. 10–12 at some specific time steps. In Fig. 10, four sources are estimated at 91, 97, 193, and 199 with strengths 1, 2.5, 1.5, and 2, respectively, when $\sigma = 0$. In Fig. 11, one source is estimated at grid point 145 with strength 7 when $\sigma = 0$. In Fig. 12, two pairs of coincided sources are estimated at grid points 138 and 152 with strengths 2.5 and 4.5, respectively, when $\sigma = 0$. When the measurement error is considered, the estimated result deviates from the exact one. When $\sigma = 0.0005$, the results in Figs. 10–12 show that the estimated locations are close to the exact locations.

In this example, the computed results also show that the exact solution cannot attach at two positions but that it is close to the exact locations. One is the sources at time steps 31–35, that is, at grid points 127, 161, 163, and 129, and the other is at the time steps 76–80, that is, at grid points 121, 155, 169, and 135. For example, four estimated sources with significant strengths are at grid points 146, 144, 145, and 112 that are close to the exact locations, that is, at grid points 127, 161, 163, and 129, in Fig. 13.

In this example, the measurement temperature errors are set within -0.001288–0.001288, which implies that the average standard deviation of measurements is 0.0005 for a 99% confidence bound. The value of relative errors is 0.000059 when $\sigma = 0.0005$.

From the results for examples 1 and 2, it is evident that there are similar trends when there is no measurement error. However,

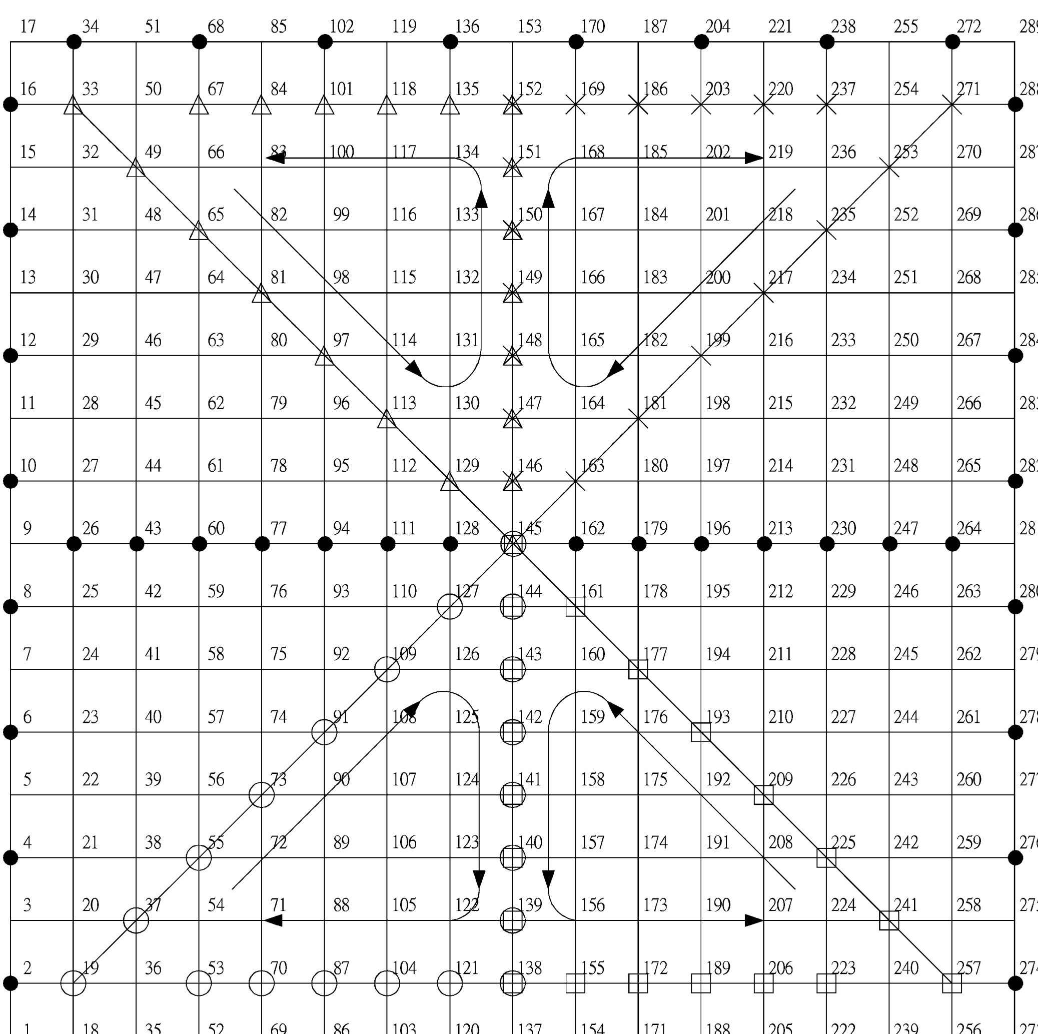

Fig. 7 Paths of four moving sources in example 2: ●, sensors; ○, source 1; □, source 2; △, source 3; and ×, source 4.

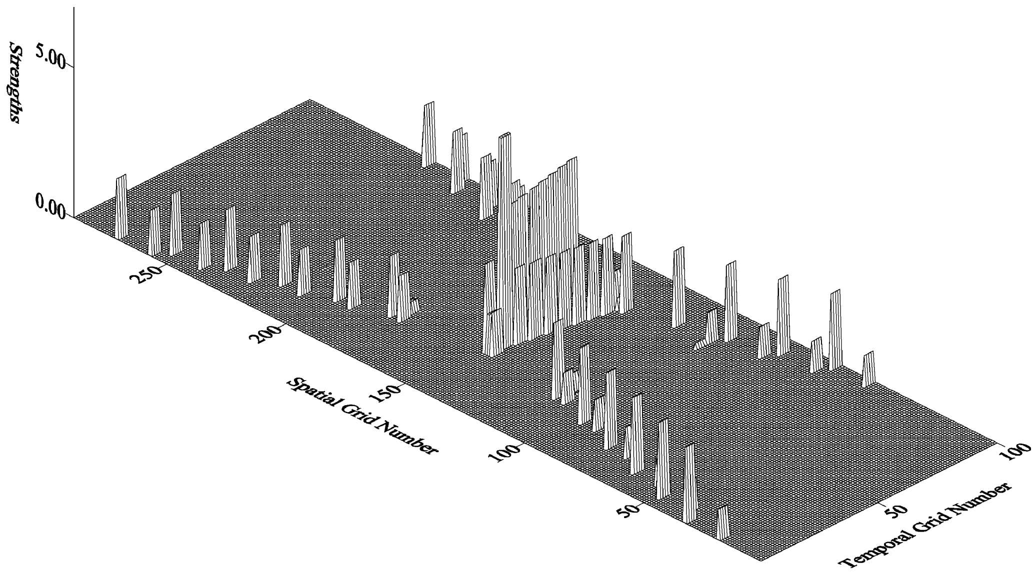

Fig. 8 Estimated results of example 2 when $\sigma = 0$.

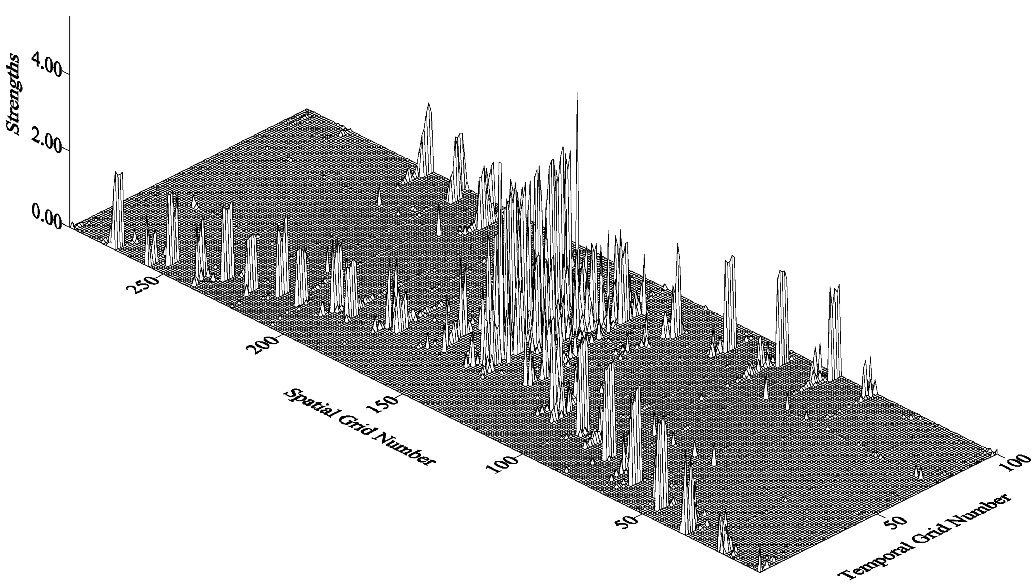

Fig. 9 Estimated results of example 2 when $\sigma = 0.0005$.

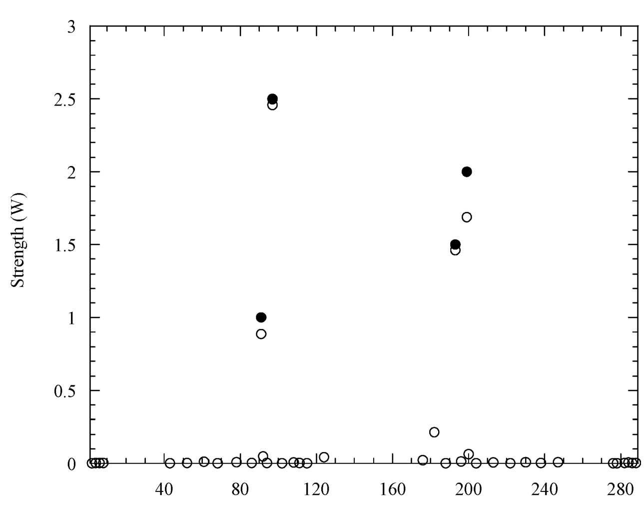

Fig. 10 Estimated results of example 2 at time step 24: ●, exact and $\sigma = 0$ and ○, $\sigma = 0.0005$.

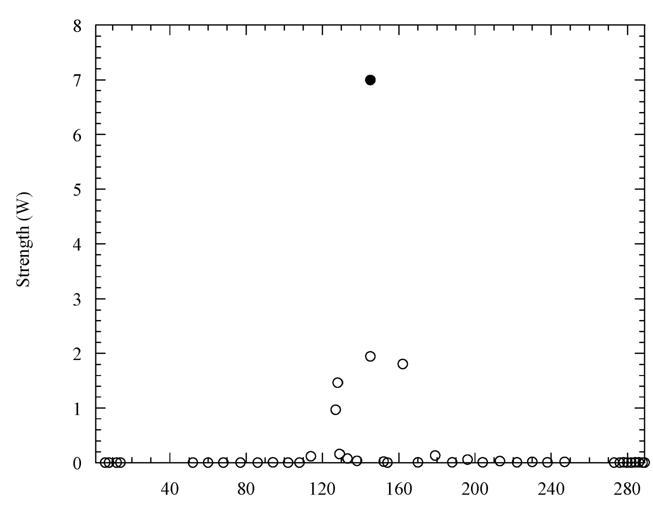

Fig. 11 Estimated results of example 2 at time step 40: ●, exact and $\sigma = 0$ and ○, $\sigma = 0.0005$.

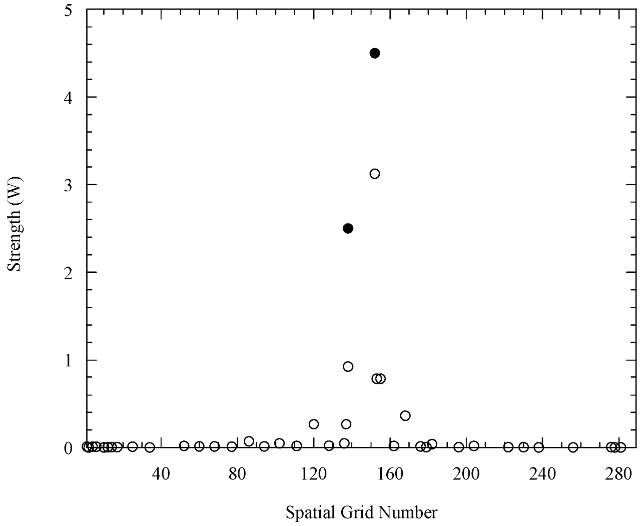

Fig. 12 Estimated results of example 2 at time step 72: ●, exact and $\sigma = 0$ and ○, $\sigma = 0.0005$.

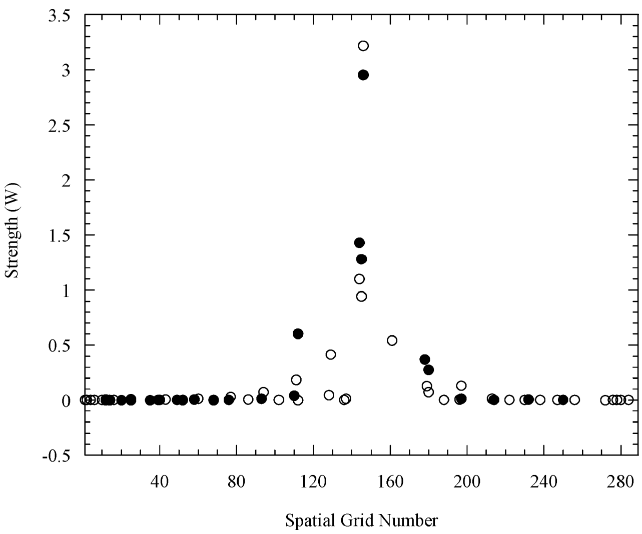

Fig. 13 Estimated results of example two at time step 32: ●, exact and $\sigma = 0$ and ○, $\sigma = 0.0005$.

the difference becomes significant when the measurement error is considered. Generally speaking, the value of the relative error increases when the number of estimated sources is increased. The numerical results show that the proposed method is robust and stable when the measurement error is included in the estimation.

Conclusions

An inverse method has been introduced for determining the location and the strength of multiple moving heat sources in two-dimensional conduction problems. The proposed inverse model is constructed from the available temperature measurements and the finite element difference model of the differential heat conduction equation. Two examples have been illustrated using the proposed method. In the first example, the results show the accuracy of the two moving source estimations. In the second example, the location and the strength of four moving sources can be identified. The proposed method solves the unknown sources sequentially, and the prior knowledge of the source number, source location, and source strength are unnecessary. The location and the strength of multiple moving sources can be determined step by step along with the temporal coordinate.

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